

On the quasi-depth of squarefree monomial ideals and the sdepth of the monomial ideal of independent sets of a graph

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Abstract

If $J \subset I$ are two monomial ideals, we give a practical upper bound for the Stanley depth of J/I , which we call it the *quasi-depth* of J/I . We compute the quasi-depth of several classes of square free monomial ideals. Also, we study the Stanley depth of the monomial ideal associated to the independent sets of a graph.

Keywords: Stanley depth, monomial ideal, independent sets of a graph.

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Introduction

Let K be a field and $S = K[x_1, \dots, x_n]$ the polynomial ring over K . Let M be a \mathbb{Z}^n -graded S -module. A *Stanley decomposition* of M is a direct sum $\mathcal{D} : M = \bigoplus_{i=1}^r m_i K[Z_i]$ as a \mathbb{Z}^n -graded K -vector space, where $m_i \in M$ is homogeneous with respect to \mathbb{Z}^n -grading, $Z_i \subset \{x_1, \dots, x_n\}$ such that $m_i K[Z_i] = \{um_i : u \in K[Z_i]\} \subset M$ is a free $K[Z_i]$ -submodule of M . We define $\text{sdepth}(\mathcal{D}) = \min_{i=1, \dots, r} |Z_i|$ and $\text{sdepth}_S(M) = \max\{\text{sdepth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M\}$. The number $\text{sdepth}_S(M)$ is called the *Stanley depth* of M . In [1], J. Apel restated a conjecture firstly given by Stanley in [16], namely that $\text{sdepth}_S(M) \geq \text{depth}_S(M)$ for any \mathbb{Z}^n -graded S -module M . This conjecture proves to be false, in general, for $M = S/I$ and $M = J/I$, where $I \subset J \subset S$ are monomial ideals, see [9].

Herzog, Vladioiu and Zheng show in [10] that $\text{sdepth}_S(M)$ can be computed in a finite number of steps if $M = I/J$, where $J \subset I \subset S$ are monomial ideals. However, it is difficult to compute this invariant, even in some very particular cases. In [17], Rinaldo give a computer implementation for this algorithm, in the computer algebra system *CoCoA* [8]. However, it is difficult to compute this invariant, even in some very particular cases. For instance in [2] Biro et al. proved that $\text{sdepth}(m) = \lceil n/2 \rceil$ where $m = (x_1, \dots, x_n)$.

In the first section, we give an upper bound for the $\text{sdepth}(J/I)$, where $I \subset J \subset S$ are two squarefree monomial ideals, in numerical terms of the associated poset $\mathcal{P}_{J/I}$, see Proposition 1.1 and Corollary 1.2. We call this bound, the quasi-depth of J/I , and we denoted by $\text{qdepth}(J/I)$. In Theorem 1.10, we give a sharp upper bound for the $\text{sdepth}(S/I)$, where $I \subset S$ is a monomial ideal generated by squarefree monomials of degree m . Our method can be useful in many cases. We consider the examples given by Duval, which disprove the Stanley Conjecture, see Example 1.4 and 1.5. In the second section, where we consider monomial ideals of independent sets associated to graphs. We give effective bounds for the sdepth of quotient ring of these ideals, see Theorem 2.4 and Corollary 2.6. We also propose an explicit formula, see Conjecture 2.7.

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1 Main results.

We recall a construction firstly presented by us in [3]. Let $\mathcal{P} \subset 2^{[n]}$. If $F, G \in \mathcal{P}$ with $F \subset G$, the interval $[F, G]$ is the set $\{H \subset \mathcal{P} : F \subset H \subset G\}$. Let $\mathbf{P} : \mathcal{P} = \bigcup_{i=1}^r [F_i, G_i]$ be a partition of \mathcal{P} , i.e. $[F_i, G_i]$ are pairwise disjoint intervals. We define the *Stanley depth* of \mathbf{P} , the number $\text{sdepth}(\mathbf{P}) = \min_{i=1}^r \{|G_i|\}$. We define the Stanley depth of \mathcal{P} , the number $\text{sdepth}(\mathcal{P}) = \max\{\text{sdepth}(\mathbf{P}) : \mathbf{P} \text{ is a partition of } \mathcal{P}\}$.

For each $0 \leq k \leq n$, we denote $\mathcal{P}_k = \{A \in \mathcal{P} : |A| = k\}$ and $\alpha_k = |\mathcal{P}_k|$. Assume that $\text{sdepth}(\mathcal{P}) = d$. It follows that \mathcal{P} admits a partition $\mathcal{P} = \bigcup_{i=1}^r [F_i, G_i]$ with $|G_i| \geq d$ for all i . Moreover, by [3, Proposition 2.2], we can assume that there exists some $1 \leq q \leq r$ such that $|G_j| = d$ for all $j \leq q$ and $|F_j| \geq d$ for all $q < j \leq r$. See also, [17, Theorem 3.5].

We define $\beta_0 = \alpha_0$, $\beta_1 = \alpha_1 - \beta_0 \binom{d}{1}$ and $\beta_k = \alpha_k - \beta_0 \binom{d}{k} - \beta_1 \binom{d-1}{k-1} - \dots - \beta_{k-1} \binom{d-k}{1}$, for all $1 \leq k \leq d$. By the proof of [3, Theorem 2.4], we get $\beta_k = |\{i : |F_i| = k\}|$, for all $0 \leq k \leq d$. Obviously, the β_k 's are non-negative. Thus, we have the following:

Proposition 1.1. ([3, Theorem 2.4]) *If $\mathcal{P} \subset 2^{[n]}$ with $\text{sdepth}(\mathcal{P}) \geq d$, then $\beta_k \geq 0$, for all $0 \leq k \leq d$.*

Note that the definition of β_k 's do not depend on the partition of \mathcal{P} . Let $\mathcal{P} \subset 2^{[n]}$ be an arbitrary poset and let $0 \leq d \leq n$ be an integer. We say that the *quasi-depth* of \mathcal{P} is at least d , and we write $\text{qdepth}(\mathcal{P}) \geq d$, if and only if, by definition, the numbers β_0, \dots, β_d , defined as above, are non-negative. We say that $\text{qdepth}(\mathcal{P}) = d$ if $d \leq n$ is the largest integer with the property that β_0, \dots, β_d are non-negative. As a direct consequence of Proposition 1.1, we get:

Corollary 1.2. $\text{sdepth}(\mathcal{P}) \leq \text{qdepth}(\mathcal{P})$.

Of course, the computation of the invariant $\text{qdepth}(\mathcal{P})$ is much easier than the computation of $\text{sdepth}(\mathcal{P})$, both from a theoretical point of view and, also, practical. Similar techniques were presented in [12, Section 7.2.1] and [11, Section 5.2], with other terminology and in a more general context.

We recall the method of Herzog, Vladoiu and Zheng [10] for computing the Stanley depth of S/I and I , where I is a squarefree monomial ideal. Let $G(I) = \{u_1, \dots, u_s\}$ be the set of minimal monomial generators of I . We define the following two posets:

$$\mathcal{P}_I := \{C \subset [n] : \text{supp}(u_i) \subset C \text{ for some } i\} \text{ and } \mathcal{P}_{S/I} := 2^{[n]} \setminus \mathcal{P}_I.$$

Also, if $I \subset J$ are two squarefree monomials ideals, we define $\mathcal{P}_{J/I} := \mathcal{P}_J \cap \mathcal{P}_{S/I}$. Herzog Vladoiu and Zheng proved in [10] that $\text{sdepth}(J/I) = \text{sdepth}(\mathcal{P}_{J/I})$. We define the *quasi-depth* of J/I , the number $\text{qdepth}(J/I) := \text{qdepth}(\mathcal{P}_{J/I})$.

Remark 1.3. Let $I \subset J \subset S$ be two squarefree monomial ideals, and let $\mathcal{P} = \mathcal{P}_{J/I}$. As above, we denote $\mathcal{P}_k = \{A \in \mathcal{P} : |A| = k\}$ and $\alpha_k = |\mathcal{P}_k|$. Let $\bar{J} = (J, x_1^2, \dots, x_n^2)$ and $\bar{I} = (I, x_1^2, \dots, x_n^2)$. We claim that $\alpha_j = \dim_K(\bar{J}/\bar{I})_j = H_{\bar{J}/\bar{I}}(j)$, for all $0 \leq j \leq n$, where $H_{\bar{J}/\bar{I}}(-)$ is the *Hilbert function* of \bar{J}/\bar{I} .

Indeed, a monomial u is in $\bar{J} \setminus \bar{I}$, if and only if u is square free and $u \in J \setminus I$. Also, the sets of \mathcal{P}_k are in bijection with the square free monomials of degree k from $J \setminus I$. Thus we proved the claim. In particular, if $J = S$, then $\alpha_j = H_{S/I}(j)$. Also, if $I = (0)$, then $\alpha_j = H_{\bar{J}}(j)$.

Example 1.4. We consider the ideal $I = (x_{13}x_{16}, x_{12}x_{16}, x_{11}x_{16}, x_{10}x_{16}, x_9x_{16}, x_8x_{16}, x_6x_{16}, x_3x_{16}, x_1x_{16}, x_{13}x_{15}, x_{12}x_{15}, x_{11}x_{15}, x_{10}x_{15}, x_9x_{15}, x_8x_{15}, x_3x_{15}, x_{13}x_{14}, x_{12}x_{14}, x_{11}x_{14}, x_{10}x_{14}, x_9x_{14}, x_8x_{14}, x_{10}x_{13}, x_9x_{13}, x_8x_{13}, x_6x_{13}, x_3x_{13}, x_1x_{13}, x_{10}x_{12}, x_9x_{12}, x_8x_{12}, x_3x_{12}, x_{10}x_{11}, x_9x_{11}, x_8x_{11}, x_6x_{10}, x_3x_{10}, x_1x_{10}, x_3x_9, x_5x_7, x_3x_7, x_2x_7, x_1x_7, x_5x_6, x_2x_6, x_1x_6, x_4x_5, x_3x_5, x_1x_4, x_4x_{15}x_{16}, x_2x_{15}x_{16}, x_2x_4x_{15}, x_6x_7x_{14}, x_1x_5x_{14}, x_4x_{12}x_{13}, x_2x_{12}x_{13}, x_2x_4x_{12}, x_6x_7x_{11}, x_1x_5x_{11}, x_4x_9x_{10}, x_2x_9x_{10}, x_2x_4x_9, x_6x_7x_8, x_1x_5x_8) \subset S = K[x_1, \dots, x_{16}]$.

According to [9, Theorem 3.5], it follows that $\text{sdepth}(S/I) < \text{depth}(S/I)$. The explicit computation of $\text{sdepth}(S/I)$ is too hard for any computer. However, if we consider the poset $\mathcal{P} = \mathcal{P}_{S/I}$, one can easily check that $\alpha_0 = 1$, $\alpha_1 = 15$, $\alpha_2 = 71$, $\alpha_3 = 98$ and $\alpha_4 = 42$. For $d = 4$, we have $\beta_0 = 1$, $\beta_1 = \alpha_1 - 4\beta_0 = 12$, $\beta_2 = \alpha_2 - \binom{4}{2}\beta_0 - \binom{3}{1}\beta_1 = 72 - 6 - 36 = 30$ and $\beta_3 = \alpha_3 - 4\beta_0 - 3\beta_1 - 2\beta_2 = 98 - 4 - 36 - 60 = -2 < 0$. Therefore, $\text{qdepth}(S/I) < 4 = \text{depth}(S/I) = \dim(S/I)$. Now, let $\mathcal{P} = \mathcal{P}_I$. It follows that $\alpha_0 = \alpha_1 = 0$, $\alpha_2 = \binom{16}{2} - 71 = 49$, $\alpha_3 = \binom{16}{3} - 98 = 462$, $\alpha_4 = \binom{16}{4} - 42 = 1778$ and $\alpha_k = \binom{16}{k}$ for all $5 \leq k \leq 16$. One can easily check that $\text{qdepth}(I) \geq 5$ so, we can expect that $\text{sdepth}(I) \geq \text{depth}(I) = 5$. Note that, the Stalney conjecture is still open in the case of ideals.

Example 1.5. Let $S = K[x_1, \dots, x_6]$, $I = (x_1x_4x_5, x_4x_6, x_2x_3x_6) \subset S$ and $J = (x_1x_2, x_1x_5, x_1x_6, x_2x_3, x_2x_4, x_4x_6) \subset S$. According to [9, Remark 3.6], we have $\text{sdepth}(J/I) = 3 < \text{depth}(J/I) = 4$. However, if we denote $\mathcal{P} = \mathcal{P}_{J/I}$, we have $\alpha_0 = \alpha_1 = 0$, $\alpha_2 = 5$, $\alpha_3 = 10$ and $\alpha_4 = 5$. By straightforward computations, we get $\text{qdepth}(J/I) = 4$.

Proposition 1.6. If $\mathbf{m} = (x_1, \dots, x_n)$, then $\text{qdepth}(\mathbf{m}) = \lceil \frac{n}{2} \rceil$.

Proof. Since $\text{sdepth}(\mathbf{m}) = \lceil \frac{n}{2} \rceil$, see [2, Theorem 2.2], it is enough to prove that for $d := \lceil \frac{n}{2} \rceil + 1$, the numbers β_k 's are not all non-negative. Let $\mathcal{P} = \mathcal{P}_{\mathbf{m}}$. We have $\alpha_0 = 0$ and $\alpha_k = \binom{n}{k}$ for all $1 \leq k \leq n$. We get, $\beta_0 = 0$, $\beta_1 = \alpha_1 = n$. Therefore $\beta_2 = \alpha_2 - \beta_1 \binom{d-1}{1} = \binom{n}{2} - n(d-1) = \frac{n}{2}(n-1-2\lceil \frac{n}{2} \rceil) < 0$ and thus we are done. \square

Proposition 1.7. (See [6, Theorem 1.1]) Let $I_{n,m}$ be the monomial ideal generated by all the square free monomials of degree m . Then $\text{sdepth}(I_{n,m}) \leq \text{qdepth}(I_{n,m}) \leq \lceil \frac{n-m}{m+1} \rceil + m - 1$.

Proof. Let $0 \leq d \leq n$ be an integer. Note that $\alpha_0 = \alpha_1 = \dots = \alpha_{m-1} = 0$ and $\alpha_k = \binom{n}{k}$ for all $m \leq k \leq n$. It follows that $\beta_0 = \beta_1 = \dots = \beta_{m-1} = 1$ and $\beta_m = \alpha_m = \binom{n}{m}$. Therefore $\beta_{m+1} = \alpha_{m+1} - d\beta_m = \binom{n}{m+1} - d\binom{n}{m}$. One can easily check that if $d \geq \lceil \frac{n-m}{m+1} \rceil + m$, then $\beta_{m+1} < 0$. Therefore, $\text{qdepth}(I_{n,m}) \leq \lceil \frac{n-m}{m+1} \rceil + m - 1$. \square

Lemma 1.8. Let $n \geq 2$, $1 \leq d \leq n$ be two integers. Let $\alpha_k = \binom{n}{k}$, for all $0 \leq k \leq d$. Let $\beta_0 = \alpha_0$ and $\beta_k = \alpha_k - \beta_0 \binom{d}{k} - \beta_1 \binom{d-1}{k-1} - \dots - \beta_{k-1} \binom{d-k}{1}$, for all $1 \leq k \leq d$. Then $\beta_k = \binom{n+k-d-1}{k}$, for all $k \leq n$.

Proof. In order to prove the Lemma, one has to check the identity:

$$\binom{n}{k} = \binom{n-d}{0} \binom{d}{k} + \binom{n-d+1}{1} \binom{d-1}{k-1} + \cdots + \binom{n+k-d-1}{k} \binom{d-k+1}{0},$$

for all $1 \leq k \leq d$, which can be done by identifying the coefficients of t^k in the identity $(1+t)^n = (1+t)^{n-d}(1+t)^d$. \square

Lemma 1.9. *Let $\mathcal{P} = \mathcal{P}' \cup \mathcal{P}'' \subset 2^{[n]}$ such that $\mathcal{P}' \cap \mathcal{P}'' = \emptyset$. Then $\text{sdepth}(\mathcal{P}) \geq \min\{\text{sdepth}(\mathcal{P}'), \text{sdepth}(\mathcal{P}'')\}$.*

Proof. It is enough to notice that, given two partitions of \mathcal{P}' and \mathcal{P}'' , one can obtain a partition of \mathcal{P} as the union of them. \square

Theorem 1.10. *Let $I \subset S$ be a squarefree monomial ideal generated in degree $m < n$ with $g = |G(I)|$.*

- (a) *If $\binom{n+m-d-1}{m} < g$, then $\text{qdepth}(S/I) \leq d-1$. In particular, $\text{sdepth}(S/I) \leq d-1$.*
- (b) *In particular, if $\binom{n-1}{m} < g$, then $\text{qdepth}(S/I) = \text{sdepth}(S/I) = m-1$.*

Proof. (a) Let $\mathcal{P} = \mathcal{P}_{S/I}$. With the notations used in the first part of the section, we have $\alpha_k = \binom{n}{k}$, for all $0 \leq k \leq m-1$, and $\alpha_m = \binom{n}{m} - g$. By Lemma 1.8, we get $\beta_k = \binom{n+k-d-1}{k}$ for all $0 \leq k < m$ and $\beta_m = \binom{n+m-d-1}{m} - g$. Therefore, if $\beta_m < 0$, it follows that $\text{qdepth}(S/I) \leq d-1$ and thus the required conclusion follows.

(b) By (a), we get $\text{sdepth}(S/I) \leq \text{qdepth}(S/I) \leq m-1$. In order to prove the converse, it is enough to notice that we can find partitions \mathbf{P}_k of the poset $2^{[n]}$ with $\text{sdepth}(\mathbf{P}_k) = k$ for any $0 \leq k \leq n$. We can write $\mathcal{P} := \mathcal{P}_{S/I} = \mathcal{P}' \cup \mathcal{P}''$, where $\mathcal{P}' = \{F \subset [n] : |F| < m\}$ and $\mathcal{P}'' = \{F \in \mathcal{P} : |F| \geq m\}$. From the previous remark, $\text{sdepth}(\mathcal{P}') = m-1$. On the other hand, $\text{sdepth}(\mathcal{P}'') \geq m$. It follows, by Lemma 1.9, that $\text{sdepth}(\mathcal{P}) \geq m-1$, and thus $\text{sdepth}(S/I) \geq m-1$. \square

In [14], D. Popescu proved a similar result for $\text{sdepth}(I)$, namely, if $\binom{n-1}{m} < g$ then $\text{sdepth}(I) = \text{depth}(I) = m$, see [14, Remark 2.7].

2 Monomial ideal of independent sets of a graph

Let $n \geq 3$ be an integer and let $G = (V, E)$ be a graph with the vertex set $V = [n]$ and edge set E . A set of vertices S is *independent* if there are no elements i and j of S such that $\{i, j\} \in E$. We denote by $\text{Ind}(G)$ the set of all the independent sets of G . Let $\alpha(G)$ be the maximal cardinality of an independent set of G , called the *independence number* of the graph G . Let $T := K[s_i, t_i : i \in [n]]$ be the ring of polynomials in two sets of n variables. For any $S \in \text{Ind}(G)$, we consider the monomial $m_S := \prod_{i \in S} s_i \prod_{i \notin S} t_i$. Let $I := (m_S : S \in \text{Ind}(G)) \subset T$. The ideal I is called the monomial ideal of independent sets associated to the graph G . The algebraic invariants of I were studied by Olteanu in [13]. Later on, Cook II studied some generalization of these ideals, see [7]. We recall several results regarding the monomial ideal of independent sets.

For any monomial $m \in G(I)$ with $m = m_S$, where $S \in \text{Ind}(G)$, we denote $m^{(s)} = \prod_{j \in S} s_j$ and $m^{(t)} = \prod_{j \notin S} t_j$ the s -part, respectively the t -part of m . Moreover, we denote $\deg_s(m) = \deg(m^{(s)})$ and $\deg_t(m) = \deg(m^{(t)})$. We consider the lexicographic order on $K[s_1, \dots, s_n]$ induced by $s_1 > s_2 > \dots > s_n$. Next, we define the monomial order \succ on the monomials of T , given by: $m \succ m'$ if and only if $\deg_s(m) < \deg_s(m')$ or, $\deg_s(m) = \deg_s(m')$ and $m^{(s)} >_{lex} m'^{(s)}$. We assume that $G(I) = \{m_1, \dots, m_r\}$, where $m_1 \succ m_2 \succ \dots \succ m_r$. Assume $m_i = m_{S_i}$ for $S_i \in \text{Ind}(G)$. Let $I_i := (m_1, \dots, m_i)$ for all $i \in [r]$. Let a_k be the number of the number of independent sets with k elements of the graph G . With this notations, we have the following result.

Theorem 2.1. ([13, Theorem 2.2], [13, Corollary 2.3]) $(I_{i-1} : m_i) = (t_r \mid r \in S_i)$, for all $i > 1$. In particular, I has linear quotients and thus linear resolution. Moreover:

- (a) $\text{reg}(I) = n$.
- (b) $\beta_i(I) = \sum_{k=0}^{\alpha(G)} a_k \binom{k}{i}$.
- (c) $\text{pd}(T/I) = \alpha(G) + 1$.
- (d) $\dim(T/I) = 2n - 2$.
- (e) $\text{depth}(T/I) = 2n - \alpha(G) - 1$.
- (f) T/I is Cohen-Macaulay if and only if G is the complete graph.

In [15], Asia Rauf proved the following result:

Lemma 2.2. Let $0 \rightarrow U \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of \mathbb{Z}^n -graded S -modules. Then $\text{sdepth}(M) \geq \min\{\text{sdepth}(U), \text{sdepth}(N)\}$.

Remark 2.3. Let \mathcal{P} be a set of squarefree monomials from T . If $u, v \in T$ with $u|v$ are two monomials, we denote $[u, v] := \{w \text{ monomial} : u|w \text{ and } w|v\}$. Let $\mathbf{P} : \mathcal{P} = \bigcup_{i=1}^r [u_i, v_i]$ be a partition of \mathcal{P} . We denote $\text{sdepth}(\mathbf{P}) := \min_{i \in [r]} \deg(v_i)$. Also, we define the Stanley depth of \mathcal{P} , to be the number $\text{sdepth}(\mathcal{P}) = \max\{\text{sdepth}(\mathbf{P}) : \mathbf{P} \text{ is a partition of } \mathcal{P}\}$.

We define the following two posets:

$$\mathcal{P}_I := \{m \in T \text{ square free} : u_i | m \text{ for some } i\} \text{ and}$$

$$\mathcal{P}_{T/I} = \{m \in T \text{ square free} : u_i \nmid m \text{ for all } i\}$$

Herzog Vladioiu and Zheng proved in [10] that $\text{sdepth}(I) = \text{sdepth}(\mathcal{P}_I)$ and $\text{sdepth}(T/I) = \text{sdepth}(\mathcal{P}_{T/I})$. Let $\mathcal{P} = \mathcal{P}_{T/I}$. Now, for $d \in \mathbb{N}$ and $u \in \mathcal{P}$, we denote

$$\mathcal{P}_d = \{u \in \mathcal{P} : \deg(u) = d\}, \quad \mathcal{P}_{d,u} = \{u \in \mathcal{P}_d : u|v\}.$$

Note that if $u \in \mathcal{P}$ such that $\mathcal{P}_{d,u} = \emptyset$, then $\text{sdepth}(\mathcal{P}) < d$. Indeed, let $\mathbf{P} : \mathcal{P} = \bigcup_{i=1}^r [u_i, v_i]$ be a partition of \mathcal{P} with $\text{sdepth}(\mathcal{P}) = \text{sdepth}(\mathbf{P})$. Since $u \in \mathcal{P}$, it follows that $u \in [u_i, v_i]$ for some i . If $\deg(v_i) \geq d$, then it follows that $\mathcal{P}_{d,u} \neq \emptyset$, since there are monomials in the interval $[u_i, v_i]$ with degree d which are divisible by u , a contradiction. Thus, $\deg(v_i) < d$ and therefore $\text{sdepth}(\mathcal{P}) < d$.

Take $u = s_1 \cdots s_{n-1} t_1 \cdots t_{n-1} \notin I$. Since $s_n u \in I$ and $t_n u \in I$, it follows that $\mathcal{P}_{u, 2n-1} = \emptyset$ and therefore $\text{sdepth}(T/I) \leq 2n - 2 = \dim(T/I)$. Of course, this inequality could be also deduced, more directly, from the fact that I is not a principal ideal.

The main result of this section is the following Theorem.

Theorem 2.4. $\text{sdepth}(T/I) \geq \text{depth}(T/I)$.

Proof. We use induction on $n \geq 1$. If $n = 1$, then $I = (s_1, t_1)$ and thus $\text{sdepth}(T/I) = \text{depth}(T/I) = 2$. Assume $n \geq 2$. We consider the short exact sequence:

$$0 \rightarrow T/(I : t_n) \rightarrow T/I \rightarrow T/(I, t_n) \rightarrow 0.$$

Let $T' = K[t_1, \dots, t_{n-1}, s_1, \dots, s_{n-1}]$ and let $J \subset T'$ be the ideal of the independent sets of the graph $G' = G \setminus \{n\}$. Note that $\alpha(G') \leq \alpha(G)$. Also $(I : t_n) = JT$ and therefore, according to [10, Lemma 3.6], $\text{sdepth}(T/(I : t_n)) = \text{sdepth}_{T'}(T'/J) \geq 2 + \text{depth}_{T'}(T'/J) = 2 + 2(n-1) - \alpha(G') - 1 = 2n - \alpha(G') - 1 \geq 2n - \alpha(G) - 1$.

On the other hand, (I, t_n) is generated by t_n and all the monomials of the form m_S with $S \in \text{Ind}(G)$ and $n \in S$. Let G'' be the graph obtained from G by deleting the vertex $\{n\}$ and all the vertices adjacent to $\{n\}$. For convenience, we may assume that G'' is a graph on the vertex set $[m]$ with $m < n$. Note that $\text{Ind}(G'') = \{S \setminus \{n\} : S \in \text{Ind}(G), n \in S\}$. Thus, $\alpha(G'') \leq \alpha(G)$ and, moreover, $(I, t_n) = (t_n, s_n L)$, where L is the ideal of independent sets of G'' . Using the induction hypothesis, [10, Lemma 3.6] and [4, Theorem 1.4], we get $\text{sdepth}(T/(I, t_n)) = 1 + 2m - 1 - \alpha(G'') + 2(n - m - 1) = 2n - 2 - \alpha(G'') \geq 2n - 1 - \alpha(G)$. Therefore, by Lemma 2.2 we are done. \square

Corollary 2.5. *If G is the complete graph, then $\text{sdepth}(T/I) = \text{depth}(T/I) = 2n - 2$.*

Proof. Since G is the complete graph, we $\alpha(G) = 1$ and, moreover:

$$I = (t_1 \cdots t_n, s_1 t_2 \cdots t_n, \dots, s_n t_1 \cdots t_{n-1}).$$

According to Theorem 2.1 and Theorem 2.4, it follows that $\text{sdepth}(T/I) \geq \text{depth}(T/I) = 2n - 2$. On the other hand, by Remark 2.3, we have $\text{sdepth}(T/I) \leq 2n - 2$, and thus we are done. \square

Let $g := |G(I)|$ the number of minimal monomial generators of I . Note that $|G(I)| \leq \sum_{i=0}^{\alpha(G)} \binom{n}{i}$. We denote $\gamma(G) = \max\{d : \binom{3n-d-1}{n} \geq g\}$. Note that $n - 1 \leq \gamma(G) \leq 2n - 2$. Indeed, for $d = n - 1$, we have $|G(I)| \leq 2^n$ and $\binom{3n-d-1}{n} = \binom{2n}{n} \geq 2^n$ for all $n \geq 1$. On the other hand, if $d > 2n - 2$, then $3n - d - 1 < n$ and therefore $\binom{3n-d-1}{n} = 0 < 1 \leq |G(I)|$.

As a direct consequence of Theorem 1.10 and Theorem 2.4 we get the following corollary.

Corollary 2.6. $\text{depth}(T/I) \leq \text{sdepth}(T/I) \leq \gamma(G) \leq 2n - 2$.

Our computer experimentations (see [8] and [17]) lead us to the following conjecture:

Conjecture 2.7. $\text{sdepth}(T/I) = \gamma(G)$.

Remark 2.8. Note that if I is minimally generated by $g > \binom{2n-1}{n}$ squarefree monomials of degree n , Theorem 1.10(b) implies $\text{sdepth}(T/I) = n - 1$. So, the conjecture holds in this very particular case. This is the case, for instance, when G is the discrete graph on $n \leq 3$ vertices. Also, by Corollary 2.5, the conjecture holds for G the complete graph.

Moreover, if $\gamma(G) = 2n - \alpha(G) - 1$, then, by Corollary 2.6, it follows that $\text{sdepth}(T/I) = \text{depth}(T/I)$. Note that $\gamma(G) = 2n - \alpha(G) - 1$ is equivalent to $\binom{n+\alpha(G)-1}{n} < g$. This is a special condition for a graph G , which holds rarely.

Example 2.9. Let $G = (V, E)$ be the cycle of length 4, i.e. $V = [4]$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$. The independent sets of G are $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}$ and $\{2, 4\}$. The associated ideal $I \subset T = K[s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4]$ of independent sets is generated by $m_1 = t_1 t_2 t_3 t_4$, $m_2 = s_1 t_2 t_3 t_4$, $m_3 = s_2 t_1 t_3 t_4$, $m_4 = s_3 t_1 t_2 t_4$, $m_5 = s_4 t_1 t_2 t_3$, $m_6 = s_1 s_3 t_2 t_4$ and $m_7 = s_2 s_4 t_1 t_3$. By Theorem 2.1, one has $\text{reg}(I) = 4$, $\text{pd}(T/I) = 3$, $\dim(T/I) = 6$ and $\text{depth}(T/I) = 5$. Moreover, $g = |G(I)| = 7$ and thus $\gamma(G) = 5$, since $\binom{6}{4} = 15 > 7$, but $\binom{5}{4} = 5 < 7$. According to Corollary 2.6, we get $\text{sdepth}(T/I) = 5$.

Let $G = (V, E)$ be the line ideal of length 5, i.e. $V = [5]$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$. Note that $\text{Ind}(G) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{1, 3, 5\}\}$. We have $\alpha(G) = 3$ and $g = |\text{Ind}(G)| = 13$. Note that $\gamma(G) = 7 > \text{depth}(T/I) = 6$, since $\binom{7}{5} = 12 > 13$ and $\binom{6}{5} = 6 < 13$.

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